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GENERALIZED MIXED VARIATIONAL PRINCIPLES AND NEW FEM MODELS IN SOLID MECHANICS

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Abstract—In this paper, generalized variational principles in the finite deformation theory of solid mechanics, called the generalized mixed variational principles (GMVP), are established. Their distinctive feature is that the functionals contain some arbitrary additional functions, called the splitting factors. GMVP brings to light more profound relations among existing variational principles, turning them into some special cases of GMVP and supplying a new mathematical foundation for numerical analysis. Applying GMVP to FEM, one can take advantage of the splitting factor to improve its precision and to circumvent some special problems, such as ill-conditioning, which often appears in non-homogeneous media, anisotropic material, incompressible bodies, singularity, penalty method and so on. In addition, this paper will give a general criterion with which to choose the splitting factor for the optimal numerical solution.

1. INTRODUCTION

Recently, a new and original type of variational principle in solid mechanics has been developed. It was first put forward in Rong (1981a, b) and applied to FEM (Rong, 1981c, d, 1985). Its distinctive feature is that the functionals contain some arbitrary additional functions, called splitting factors. Later, Qian (1983) discussed the same type of variational principles. However, they were all linear theories. The present paper will present this type of variational principle for the non-linear theory, called the generalized mixed variational principle (GMVP). The motivation to establish GMVP came from the following idea of research in FEM. As is well known, the usual FEM model, which stems from the minimum potential energy principle, is more rigid, while the complementary energy principle makes it more flexible than the real body. Therefore, it is hoped to arrive at some intermediate case, i.e. to obtain a numerical solution lying between the lower and upper bounds (Zienkiewicz, 1977). One expects it to be closer to the exact solution. To this end, one has to search for a new type of variational principle which has the following two characteristics.

I. Its functional contains both the strain energy and the complementary one.

II. The component of the strain energy (or the complementary one) can be adjusted arbitrarily.

The first requirement is apparent, otherwise one could return to the old way, while the second is also necessary. As the exact solution lies between the lower and upper bounds, it is impossible for the numerical solution to approach the exact one without a factor to adjust the contribution of the strain energy (or the complementary one) to the functional. Obviously, the desired principle must be of mixed type. Although there have been several mixed variational principles (Washizu, 1975; Qian, 1980), none of them possess both characteristics I and II together as the new type of variational principle (linear theory) presented in Rong (1981a, b) does. Characteristic II is embodied in the arbitrary splitting factor, showing the portion of the strain energy in the functional.

2. THE SPLIT FORM OF BASIC EQUATIONS IN NON-LINEAR ELASTICITY

In order to establish the new type of variational principle for the finite deformation theory, it is necessary to introduce some fresh concepts to describe the constitutive law for the non-linear elasticity. Below one will build up a new set of basic equations, which are equivalent to the traditional ones and convenient to discuss GMVP. In Cartesian coordinates x_i, $i = 1, 2, 3$, the traditional form of the basic equations for the finite deformation theory can be given by

$$[(\delta_{ki}+u_{k,i})\sigma_{ij}]_{,j}+\bar{F}_k = 0 \quad \text{in } V \tag{1}$$

$$e_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}) \quad \text{in } V \tag{2}$$

$$\sigma_{ij} = \frac{\partial A}{\partial e_{ij}} \quad \text{in } V \tag{3}$$

or

$$e_{ij} = \frac{\partial B}{\partial \sigma_{ij}} \quad \text{in } V \tag{4}$$

$$u_i = \bar{u}_i \quad \text{on } S_u \tag{5}$$

$$P_i \equiv (\delta_{ki}+u_{k,i})\sigma_{ij}n_j = \bar{P}_i \quad \text{on } S_p \tag{6}$$

where V is the body domain with its surface $S = S_u+S_p$, subjected to the body force $\bar{F}_i$. On S_u are given displacements $\bar{u}_i$ and on S_p boundary forces $\bar{P}_i$. u_i, e_{ij} and σ_{ij} denote the displacement vector, the strain tensor and the stress tensor in Cartesian coordinates, respectively; n_j is the outward unit vector normal to the surface S; δ_{ij} is the Kronecker symbol and $(\ldots)_{,j}$ denotes the derivative with respect to x_j. A and B are the strain energy density and the complementary one

$$A(e_{ij}) = \int_0^{e_{ij}} \sigma_{kl}\, \mathrm{d}e_{kl}$$

and

$$B(\sigma_{ij}) = \int_0^{\sigma_{ij}} e_{kl}\, \mathrm{d}\sigma_{kl}.$$

In the following one introduces a new form of basic equation equivalent to the above one.

Let β_{ijkl} be some arbitrary fourth-order tensor over V, called the splitting factor, which satisfies certain requirements

$$\beta_{ijkl} = \beta_{jikl} = \beta_{ijlk}. \tag{7}$$

With the help of β_{ijkl} one splits σ_{ij} into two parts

$$\sigma_{ij} = \beta_{ijkl}\sigma_{kl}+\sigma^*_{ij} \tag{8}$$

or

$$\sigma_{ij} = \beta_{ijkl}\frac{\partial A}{\partial e_{kl}}+\sigma^*_{ij} \tag{9}$$

where

$$\sigma^*_{ij} = (\delta_{ik}\delta_{jl}-\beta_{ijkl})\sigma_{kl}. \tag{10}$$

σ^*_{ij} is called the split stress tensor. More constraints are put on β_{ijkl}

$$\beta_{ijkl}\frac{\partial^2 A}{\partial e_{st}\partial e_{kl}} = \beta_{stkl}\frac{\partial^2 A}{\partial e_{ij}\,\partial e_{kl}} \tag{11}$$

and for any non-zero symmetric second-order tensor G_{ij}

$$\beta_{ijkl}G_{ij}G_{kl} \neq 0. \tag{12}$$

In addition, one introduces another fourth-order tensor α_{mnij} associated with β_{ijkl}

$$\alpha_{mnij} = \alpha_{nmij} = \alpha_{mnji} \tag{13}$$

$$\left.\frac{\partial^2 B}{\partial \sigma_{ij}\,\partial \sigma_{kl}}\right|_{\sigma=\alpha\sigma^*}\alpha_{klst} = \left.\frac{\partial^2 B}{\partial \sigma_{kl}\,\partial \sigma_{st}}\right|_{\sigma=\alpha\sigma^*}\alpha_{klij} \tag{14}$$

$$\alpha_{mnij}(\delta_{ik}\delta_{jl} - \beta_{ijkl}) = \delta_{mk}\delta_{nl} \tag{15}$$

where $\sigma = \alpha\sigma^*$ is the abbreviation for

$$\sigma_{mn} = \alpha_{mnpq}\sigma^*_{pq} \tag{16}$$

which is guaranteed by eqns (10), (13) and (15).

For the given A, B and a chosen splitting factor one can introduce two new energy density functions, denoted by A^* and B^*, as follows

$$A^* = \int_0^{e_{ij}} \beta_{mnkl}\frac{\partial A}{\partial e_{kl}}\,\mathrm{d}e_{mn} = \int_0^{e_{ij}} \beta_{mnkl}\sigma_{kl}\,\mathrm{d}e_{mn} \tag{17}$$

$$B^* = \int_0^{\sigma^*_{ij}} \left.\frac{\partial B}{\partial \sigma_{mn}}\right|_{\sigma=\alpha\sigma^*}\mathrm{d}\sigma^*_{mn} = \int_0^{\sigma^*_{ij}} e_{mn}(\alpha\sigma^*)\,\mathrm{d}\sigma^*_{mn}. \tag{18}$$

From eqns (9) and (16)–(18), one has

$$\left.\begin{aligned} e_{ij} &= \frac{\partial B^*}{\partial \sigma^*_{ij}} \\ \sigma_{ij} &= \frac{\partial A^*}{\partial e_{ij}} + \sigma^*_{ij}. \end{aligned}\right\} \tag{19}$$

Below one will show that to express the constitutive law, eqn (19) is equivalent to eqn (3) or (4). First, note that eqns (11) and (14) are the requirements to make A^* and B^* independent of the integrating paths. Therefore, for the given A, B and a chosen splitting factor, there must exist a definite A^* and B^*. Second, eqns (3) and (4) are equivalent to each other, i.e.

$$\sigma_{ij} = \frac{\partial A}{\partial e_{ij}} \Leftrightarrow e_{ij} = \frac{\partial B}{\partial \sigma_{ij}}. \tag{20}$$

Next one will show the equivalence (19) $\Leftrightarrow$ (3) or (4), i.e.

$$\left(e_{ij} = \frac{\partial B^*}{\partial \sigma^*_{ij}} \quad \text{and} \quad \sigma_{ij} = \frac{\partial A^*}{\partial e_{ij}} + \sigma^*_{ij}\right) \Leftrightarrow \left(\sigma_{ij} = \frac{\partial A}{\partial e_{ij}} \quad \text{or} \quad e_{ij} = \frac{\partial B}{\partial \sigma_{ij}}\right). \tag{21}$$

In fact, as A^* and B^* are defined by A and B, ((3) or (4)) $\Rightarrow$ (19) holds naturally. Hence, one has only to show (19) ⇒ ((3) or (4)). From the first part of eqn (19) and equivalence (20), one has

$$e_{ij} = \frac{\partial B^*}{\partial \sigma^*_{ij}} = e_{ij}(\alpha\sigma^*)$$

then

$$\alpha_{ijkl}\sigma^*_{kl} = \frac{\partial A}{\partial e_{ij}}.$$

Noting eqn (15), one has

$$\sigma^*_{ij} = (\delta_{ik}\delta_{jl} - \beta_{ijkl})\frac{\partial A}{\partial e_{kl}}.$$

Finally, from the second part of eqn (19), one obtains

$$\sigma_{ij} = \beta_{ijkl}\frac{\partial A}{\partial e_{kl}} + (\delta_{ik}\delta_{jl} - \beta_{ijkl})\frac{\partial A}{\partial e_{kl}} = \frac{\partial A}{\partial e_{ij}}.$$

Hence, equivalence (21) is proven. In addition, under the condition $\sigma^*_{ij} = (\delta_{ik}\delta_{jl} - \beta_{ijkl})\sigma_{kl}$ one has one more equivalence

$$e_{ij} = \frac{\partial B}{\partial \sigma_{ij}} \Leftrightarrow e_{ij} = \frac{\partial B^*}{\partial \sigma^*_{ij}}. \tag{22}$$

Thus, one has three options (3), (4) and (19) to describe the constitutive law. In this paper eqn (19) is used. Equations (1), (2) and (19) together with boundary conditions (5) and (6) are taken as the basic equations for the finite deformation theory and are called the split form of the basic equations. It is apparent from the above discussion that the split form is different in form from the usual one, but completely equivalent to it. The new form contains some arbitrary additional functions, which will play an important role in establishing the new models of FEM.

3. GENERALIZED MIXED VARIATIONAL PRINCIPLES

For the new set of basic equations, a new type of variational principle can be established. Its functionals may have four classes of independent function variables, defined as

$$\Pi_{4p} = \int_V [A^* + (\sigma^*_{ij} - \sigma_{ij})e_{ij} - B^* + \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - \int_{S_p} \bar{P}_k u_k\,\mathrm{d}S - \int_{S_u} (u_k - \bar{u}_k)P_k\,\mathrm{d}S \tag{23}$$

$$\Pi_{4c} = \int_V \langle(\sigma_{ij} - \sigma^*_{ij})e_{ij} - A^* + B^* + \tfrac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + \{[(\delta_{ki} + u_{k,i})\sigma_{ij}]_{,j} + \bar{F}_k\}u_k\rangle\,\mathrm{d}V - \int_{S_u} (P_k - \bar{P}_k)u_k\,\mathrm{d}S - \int_{S_u} P_k\bar{u}_k\,\mathrm{d}S = -\Pi_{4p}. \tag{24}$$

The variational principle can be stated as: among all the admissible functions of the four classes of independent variables, i.e. u_i, e_{ij}, σ_{ij} and σ^*_{ij}, the solution to the boundary problem of finite deformation theory is that which makes the functional Π_{4p} or Π_{4c} stationary. In fact, with the help of integration by parts and the Green theorem, $\delta\Pi_{4p} = 0$ or $\delta\Pi_{4c} = 0$ leads to

$$\delta\Pi_{4p} = -\delta\Pi_{4c} = \int_V \left\langle \left(\frac{\partial A^*}{\partial e_{ij}} - \sigma_{ij} + \sigma_{ij}^*\right)\delta e_{ij} + \left(e_{ij} - \frac{\partial B^*}{\partial \sigma_{ij}^*}\right)\delta\sigma_{ij}^* \right.$$

$$\left. + [\tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) - e_{ij}]\delta\sigma_{ij} - \{[(\delta_{ki} + u_{k,i})\sigma_{ij}]_{,j} + \bar{F}_k\}\,\delta u_k \right\rangle \mathrm{d}V$$

$$-\int_{S_u} (u_k - \bar{u}_k)\delta P_k \,\mathrm{d}S + \int_{S_p} (P_k - \bar{P}_k)\delta u_k \,\mathrm{d}S = 0 \quad (25)$$

which gives eqns (1), (2), (19), (15) and (6), as a result of the independence of δu_i, δe_{ij}, $\delta\sigma_{ij}$ and $\delta\sigma_{ij}^*$. It is called the generalized mixed variational principle (of four classes of independent variables). The splitting factor β_{ijkl} and the associated α_{ijkl} can be chosen arbitrarily, but must satisfy the requirements mentioned in Section 2. For simplicity, one uses the following abbreviations:

$$\left.\begin{aligned} D_{ij} &\equiv \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \\ L_k &\equiv [(\delta_{ki} + u_{k,i})\sigma_{ij}]_{,j} + \bar{F}_k \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} T_p &\equiv \int_{S_p} \bar{P}_i u_i \,\mathrm{d}S + \int_{S_u} (u_i - \bar{u}_i)P_i \,\mathrm{d}S \\ T_c &\equiv \int_{S_p} (P_i - \bar{P}_i)u_i \,\mathrm{d}S + \int_{S_u} \bar{u}_i P_i \,\mathrm{d}S. \end{aligned}\right\} \quad (27)$$

In the following, some varieties of GMVP are given by setting certain constraints among u_i, e_{ij}, σ_{ij} and σ_{ij}^* contained in Π_{4p} and Π_{4c}.

(1) $\sigma_{ij}^* = (\delta_{ik}\delta_{jl} - \beta_{ijkl})\sigma_{kl}$, i.e. eqn (10) taken as a constraint. Then one has

$$B^* = \int_0^{\sigma_{mn}} (\delta_{ik}\delta_{jl} - \beta_{ijkl})\frac{\partial B}{\partial \sigma_{ij}} \,\mathrm{d}\sigma_{kl}. \quad (28)$$

Thus, from eqns (23) and (24) one obtains the functionals for GMVP with three classes of independent variables (u_i, e_{ij} and σ_{ij})

$$\Pi_{3p} = \int_V (A^* - \beta_{ijmn}e_{ij}\sigma_{mn} - B^* + D_{ij}\sigma_{ij} - \bar{F}_k u_k) \,\mathrm{d}V - T_p \quad (29)$$

$$\Pi_{3c} = \int_V (B^* - A^* + \beta_{ijmn}e_{ij}\sigma_{mn} + \tfrac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k) \,\mathrm{d}V - T_c = -\Pi_{3p}. \quad (30)$$

Noting eqns (17), (12) and (10), the variational equations $\delta\Pi_{3p} = 0$ or $\delta\Pi_{3c} = 0$ leads to the basic equations, which are equivalent to eqns (1)–(6).

(2) $\sigma_{ij}^* = (\delta_{ik}\delta_{jl} - \beta_{ijkl})\sigma_{kl}$ and $e_{ij} = D_{ij}$. From eqns (29) and (30) one can obtain the functionals for GMVP with only two classes of independent variables (i.e. u_i and σ_{ij})

Table 1. Functional expressions

Symbols	Functionals	Splitting factors	Independent variables	Constraint conditions	Euler equations
Π_{4p}	$\int_V [A^* + (\sigma^*_{ij} - \sigma_{ij})e_{ij} - B^* + D_{ij}\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - T_p$	β_{ijkl}	$\sigma^*_{ij}, \sigma_{ij}$; e_{ij}, u_i; $u_i(s), P_i(s)$		1, 2; 3, 4; 5, 6
Π_{4c}	$\int_V [(\sigma_{ij} - \sigma^*_{ij})e_{ij} - A^* + B^* + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k]\,\mathrm{d}V - T_c$				
$\Pi_{4p\beta}$	$\int_V [\beta A + (\sigma^*_{ij} - \sigma_{ij})e_{ij} - B^* + D_{ij}\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - T_p$	β			
$\Pi_{4c\beta}$	$\int_V [(\sigma_{ij} - \sigma^*_{ij})e_i - \beta A + B^* + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k]\,\mathrm{d}V - T_c$				
Π_{3p}	$\int_V (A^* - e_{ij}\beta_{ijmn}\sigma_{mn} - B^* + D_{ij}\sigma_{ij} - \bar{F}_k u_k)\,\mathrm{d}V - T_p$	β_{ijkl}	σ_{ij}; e_{ij}; u_i; $u_i(s)$; $P_i(s)$	1	2; 3; 4; 5; 6
Π_{3c}	$\int_V (B^* - A^* + e_{ij}\beta_{ijmn}\sigma_{mn} + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
$\Pi_{3p\beta}$	$\int_V (\beta A - \beta e_{ij}\sigma_{ij} - (1-\beta)B + D_{ij}\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - T_p$	β			
$\Pi_{3c\beta}$	$\int_V [(1-\beta)B - \beta A + \beta e_{ij}\sigma_{ij} + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k]\,\mathrm{d}V - T_c$				
Π_{3p1}	$\int_V [A - (e_{ij} - D_{ij})\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - T_p$	1.0			
Π_{3c1}	$\int_V (e_{ij}\sigma_{ij} - A + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
Π_{3p0}	$\int_V (D_{ij}\sigma_{ij} - B - \bar{F}_k u_k)\,\mathrm{d}V - T_p$	0			3, 4; 5, 6
Π_{3c0}	$\int_V (B + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
Π_{2p}	$\int_V (A^* + D_{ij}\sigma^*_{ij} - B^* - \bar{F}_k u_k)\,\mathrm{d}V - T_p$	β_{ijkl}	σ_{ij}; u_i; $u_i(s)$; $P_i(s)$	1; 2	3; 4; 5; 6
Π_{2c}	$\int_V (B^* - A^* + \beta_{ijkl}\sigma_{kl}D_{ij} + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
$\Pi_{2p\beta}$	$\int_V [\beta A + (1-\beta)D_{ij}\sigma_{ij} - (1-\beta)B - \bar{F}_k u_k]\,\mathrm{d}V - T_p$	β			
$\Pi_{2c\beta}$	$\int_V [(1-\beta)B - \beta A + \beta D_{ij}\sigma_{ij} + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k]\,\mathrm{d}V - T_c$				
Π_{2p0}	$\int_V (D_{ij}\sigma_{ij} - B - \bar{F}_k u_k)\,\mathrm{d}V - T_p$	0			
Π_{2c0}	$\int_V (B + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
Π_{2p1}	$\int_V (A - \bar{F}_k u_k)\,\mathrm{d}V - T_p$	1.0			4, 5; 6
Π_{2c1}	$\int_V (D_{ij}\sigma_{ij} - A + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
Π_{1p}	$\int_V (A - \bar{F}_k u_k)\,\mathrm{d}V - T_p$		$u_i, u_i(s)$; $P_i(s)$	1, 2; 3	4, 5; 6
Π_{1c}	$\int_V (B + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c$				
Π_{0p}	$\int_V (A - \bar{F}_k u_k)\,\mathrm{d}V - T_p$		$u_i(s)$	1–5	6
Π_{0c}	$\int_V (B + \frac{1}{2}u_{k,i}u_{k,j}\sigma_{ij})\,\mathrm{d}V - T_c$		$P_i(s)$	1–4, 6	5

Note: the numbers listed in the last two columns stand for certain equations: 1 for (10), 2 for (2), 3 for (3) or (4) or $D_{ij} = \partial B/\partial\sigma_{ij}$, 4 for (1), 5 for (5), 6 for (6). $u_i(s)$ stands for u_i on the boundary S and $P_i(s)$ for P_i on S.

$$\Pi_{2p} = \int_V (A^* + e_{ij}\sigma^*_{ij} - B^* - \bar{F}_k u_k)\,\mathrm{d}V - T_p \tag{31}$$

$$\Pi_{2c} = \int_V (B^* - A^* + \beta_{ijmn}e_{ij}\sigma_{mn} + \tfrac{1}{2}u_{k,i}u_{k,j}\sigma_{ij} + L_k u_k)\,\mathrm{d}V - T_c = -\Pi_{2p}. \tag{32}$$

The variational equation $\delta\Pi_{2p} = 0$ or $\delta\Pi_{2c} = 0$ leads to the required basic equations.

(3) The splitting factor takes some special values. If β_{ijkl} takes special values, the functionals defined above may have special expressions, e.g. $\beta_{ijkl} = \beta\delta_{ik}\delta_{jl}$, where β is an arbitrary scalar function over V, and then the functionals Π_{qp} and Π_{qc} ($q = 4, 3, 2$, see Table 1) will contain only one additional function $\beta(x_i)$. If one further sets $\beta = 1$ or 0, they will give the functionals of the existing variational principles (Washizu, 1975; Qian, 1980), including the Hu–Washizu and Reissner principles (Π_{3p1} and Π_{2p0}, see Table 1).

The functionals mentioned above are all listed in Table 1, from which one can clearly see the multiplying procedure of variational principles at various levels. At each level there are two functional forms: Π_{qp} and Π_{qc}. $q = 4, 3, 2, 1, 0$, called the dual forms.

4. GMVP IN LINEAR THEORY OF ELASTICITY

In linear theory, the basic equations do not contain the terms of higher order and the expressions of the energy density functions, denoted now by A_0 and B_0, can be written explicitly as $A_0 = \frac{1}{2}S_{ijkl}e_{ij}e_{kl}$ and $B_0 = \frac{1}{2}C_{ijkl}\sigma_{ij}\sigma_{kl}$, where S_{ijkl} and C_{ijkl} are the elastic tensor and the compliant tensor, respectively. The stress–strain relations can also be given explicitly by the generalized Hooke's law

$$\sigma_{ij} = S_{ijkl}e_{kl} \tag{33}$$

or

$$e_{ij} = C_{ijkl}\sigma_{kl}. \tag{34}$$

Thus, eqns (9), (11), (14) and (17)–(19) can be rewritten, respectively, as

$$\sigma_{ij} = \beta_{ijkl}S_{klmn}e_{mn} + \sigma^*_{ij} \tag{35}$$

$$\beta_{ijkl}S_{klst} = \beta_{stkl}S_{klij} \tag{36}$$

$$C_{ijkl}\alpha_{klst} = C_{stkl}\alpha_{klij} \tag{37}$$

$$A^*_0 = \tfrac{1}{2}\beta_{mnkl}S_{klpq}e_{pq}e_{mn} \tag{38}$$

$$B^*_0 = \tfrac{1}{2}C_{mnkl}\alpha_{klpq}e_{pq}e_{mn} \tag{39}$$

$$\sigma_{ij} = \beta_{ijkl}S_{klmn}e_{mn} + \sigma^*_{ij} \quad \text{and} \quad e_{ij} = C_{ijkl}\alpha_{klmn}\sigma^*_{mn}. \tag{40}$$

Noting $P_i \equiv \sigma_{ij}n_j$ and neglecting small quantities of higher order in the functionals listed in Table 1, one can obtain the functionals of GMVP for the linear theory, denoted by Π^0_{qp} and Π^0_{qc} ($q = 4, 3, 2, 1, 0$), e.g.

$$\Pi^0_{4p} = \int_V [\tfrac{1}{2}\beta_{mnkl}S_{klpq}e_{pq}e_{mn} + e_{ij}(\sigma^*_{ij} - \sigma_{ij}) - \tfrac{1}{2}C_{mnkl}\alpha_{klpq}\sigma^*_{pq}\sigma^*_{mn} + \tfrac{1}{2}(u_{i,j} + u_{j,i})\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - T_p \tag{41}$$

$$\Pi^0_{4c} = \int_V [e_{ij}(\sigma_{ij} - \sigma^*_{ij}) - \tfrac{1}{2}\beta_{mnkl}S_{klpq}e_{pq}e_{mn} + \tfrac{1}{2}C_{mnkl}\alpha_{klpq}\sigma^*_{pq}\sigma^*_{mn} + (\sigma_{ij,j} + \bar{F}_i)u_i]\,\mathrm{d}V - T_c = -\Pi^0_{4p}. \tag{42}$$

With the four classes of independent variables, i.e. u_i, e_{ij}, σ_{ij} and σ^*_{ij}, the variational equation $\delta\Pi^0_{4p} = 0$ or $\Pi^0_{4p} = 0$ leads to the basic equations for the linear theory of elasticity. Herein the stress–strain relations are expressed by eqn (40), which is equivalent to the usual one, eqn (33) or (34).

The functionals with three classes of independent variables are

$$\Pi^0_{3p} = \int_V [\tfrac{1}{2}\beta_{mnkl}S_{klpq}e_{pq}e_{mn} - e_{ij}\beta_{ijmn}\sigma_{mn} - \tfrac{1}{2}C_{mnkl}\alpha_{klpq}\sigma^*_{pq}\sigma^*_{mn} + \tfrac{1}{2}(u_{i,j} + u_{j,i})\sigma_{ij} - \bar{F}_k u_k]\,\mathrm{d}V - T_p \tag{43}$$

$$\Pi^0_{3c} = \int_V [\tfrac{1}{2}C_{mnkl}\alpha_{klpq}\sigma^*_{pq}\sigma^*_{mn} - \tfrac{1}{2}\beta_{mnkl}S_{klpq}e_{pq}e_{mn} + e_{ij}\beta_{ijmn}\sigma_{mn} + (\sigma_{ij,j} + \bar{F}_i)u_i]\,\mathrm{d}V - T_c = -\Pi^0_{3p} \tag{44}$$

where $\sigma_{ij}^* = (\delta_{ik}\delta_{jl} - \beta_{ijkl})\sigma_{kl}$ are constraints, while the independent variables are u_i, e_{ij} and σ_{ij}. If one sets $\beta_{ijmn} = -2A_{ijkl}C_{klmn}$ or $\beta_{ijmn} = \delta_{im}\delta_{jn} + 2B_{ijkl}C_{klmn}$, where A_{ijkl} and B_{ijkl} are so-called higher-order Lagrange multipliers defined by Qian (1983), then Π_{3p}^0 and Π_{3c}^0 will turn into the functionals Π_{GI} and Π_{GII} presented in Qian (1983).

The functionals with two classes of independent variables u_i and σ_{ij} can be deduced directly from the above equations, e.g.

$$\Pi_{2p\beta}^0 = \int_V [\beta A_0 + (1-\beta)D_{ij}\sigma_{ij} - (1-\beta)B_0 - \bar{F}_i u_i] \, \mathrm{d}V - T_p \tag{45}$$

$$\Pi_{2c\beta}^0 = \int_V [(1-\beta)B_0 - \beta A_0 + \beta D_{ij}\sigma_{ij} + (\sigma_{ij,j} + \bar{F}_i)u_i] \, \mathrm{d}V - T_c = -\Pi_{2p\beta}^0. \tag{46}$$

Equation (45) is the functional presented in Rong (1981a) (see eqn (23) therein), which was called the variational principle with split modulus. Although this functional is a special case (containing only one arbitrary function β) of GMVP, it is the first example of this new type of variational principle. Its earliest applications were published in Rong (1981c, d). If one sets $\beta = 0.5$, then eqn (45) will turn into the functional given by Liang and Fu (1982).

5. A GENERAL CRITERION TO CHOOSE SPLITTING FACTORS FOR FINITE ELEMENT ANALYSIS

Every functional of GMVP established above contains an arbitrary splitting factor. It does not affect the theoretically exact solutions, because the Euler equations of these functionals are completely equivalent to the traditional basic equations. However, it does have an effect on numerical solutions by FEM and that is why it has been introduced into the new variational principles. How to choose the splitting factor to give the best numerical solutions is of great interest in practice. Below a general criterion for a reasonable choice of splitting factor is suggested, based on the following argument. The fact that the exact solution to an elasticity problem is not affected by the splitting factor can be restated mathematically as: the partial variation of any desired quantity, denoted by K (such as displacement, stress or another quantity such as stress intensity factor) with respect to the splitting factor is identically equal to zero, i.e. $\delta_\beta K \equiv 0$, where β represents any splitting factor. But in FEM, the desired quantity, denoted by $\tilde{K}$, is an approximation of K and depends on β. Thus, a general criterion is suggested as follows

$$\delta_\beta \tilde{K} = 0. \tag{47}$$

In order to carry out eqn (47) one may devise various schemes. For instance, according to the concept of FEM, one may first interpolate β in each element and obtain a functional of nodal variables of β. Then one uses eqn (47) to establish simultaneous algebraical equations. This is theoretically correct, but not convenient in practice. Below a practical approach to the implementation of eqn (47) is suggested, taking one from a large class of finite element problems as a typical example. By trial computations one can obtain an optimum value of β, say $\bar{\beta}$, which satisfies eqn (47) quite well. Then for the other problems of the same class one just uses the chosen β instead of choosing it each time. Although this is not theoretically the best solution, it is rather satisfactory in practice, as shown by the results obtained by the present author.

As an example, an approximate splitting factor of $\bar{\beta} = 0.0001$ has been chosen for the computations of stress intensity factors. The specimen is a tensile strip with an edge crack

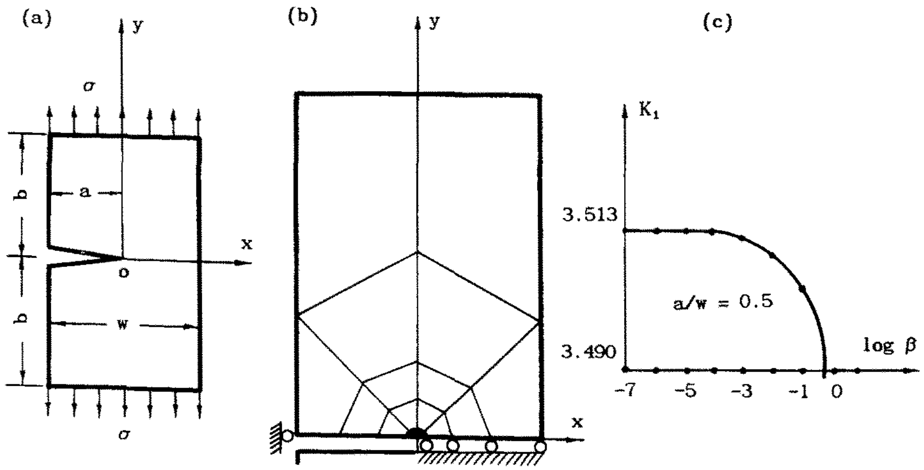

Fig. 1. Tensile strip with an edge crack for choosing the optimal splitting factor.

Table 2. Result of computing

Source of results	$a/w = 0.5$ No. of elements	K_1	Percentage error (%)	$a/w = 0.8$ No. of elements	K_1	Percentage error (%)
GMVP with $\beta = 10^{-4}$	14	3.513	−0.76	17	18.84	−2.7
Long *et al.* (1982)	14	3.60	1.7	17	18.53	−4.2
Qian *et al.* (1980)	448	3.44	−2.8	448	20.05	3.5
Benzley (1974)	110	3.50	−1.04	110	17.50	−9.7
Keer and Freedman (1973)		3.54			19.37	

as shown in Fig. 1(a) and its details can be found in Long *et al.* (1982). Eight-node isoparametric elements are used and the grid pattern is shown in Fig. 1(b). The simplest form of GMVP with $q = 2$, i.e. $\Pi^0_{2p\beta}$ (eqn (45)) is employed to establish the finite element model and the case $a/w = 0.5$ is taken for choosing β. Noting the characteristics of the stress intensity problems, one puts a constant value $\beta \neq 1$ on only a few elements around the singularity 0, while on the other elements is put $\beta = 1$ or 0 (for details see Rong (1985)). After calculation, the result is plotted as a curve of the stress intensity factor K_I vs the splitting factor β as shown in Fig. 1(c). Apparently, when β is approximately 0.0001, the requirement $\delta_\beta K_\mathrm{I} = 0$ may be considered already satisfied. Therefore, the optimum splitting factor for the case $a/w = 0.5$ is about 0.0001. When calculating stress intensity factors for the other cases, one does not have to choose β each time, but take $\beta = 0.0001$. For example, the author has also calculated K_I for the case $a/w = 0.8$ by using the same $\beta = 0.0001$. These results and those by some others are listed in Table 2 for comparison. A detailed description of this example is given in Rong (1985).

The results show that the FEM based on GMVP together with the suggested criterion gives a solution of higher precision. In addition, due to the flexible adjustability of the splitting factors, GMVP can be conveniently used to deal with some special problems, such as ill-conditioned problems, contact bodies and so on. The applications of GMVP to these problems can be found in Rong (1981c, d, 1985) and Rong and Taylor (1987). Some of the problems of GMVP require further research, e.g. "what is the difference in the similar two forms Π_{qp} and Π_{qc} ($q = 4, 3, 2$) in terms of computational efficiency?" These will be discussed by the present author elsewhere.

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